International Journal of Theoretical Physics, Vol. 16, No. 9 (1977), pp. 659–661

Remarks on Gauge Variables and Singular Lagrangians

J. CHELA-FLORES

Centro de Física, IVIC, Apdo. 1827, Caracas 101, Venezuela

R. JÁNICA-DE-LA-TORRE

Departamento de Física, Facultad de Ciencias,
Universidad Central de Venezuela, Caracas, Venezuela

A. J. KÁLNAY, J. RODRÍGUEZ-GÓMEZ, J. RODRÍGUEZ-NUÑEZ

Centro de Física, IVIC, Apdo. 1827, Caracas 101, Venezuela

and R. TASCÓN

Departamento de Matemáticas y Física
Instituto Universitario Pedagógico de Caracas,
A. Páez, El Paraíso, Caracas 102, Venezuela

Received: 26 *April* 1977

Chela-Flores (1975) has proposed a gauge theory of He-II superfluidity, based on the Lagrangian density

$$\mathscr{L}^{(k)}_{(U)} = -(i/2)(\psi\partial_t\psi^* - \psi^*\partial_t\psi) - (1/2M)[(\nabla + iM\mathbf{A})\psi^*]\cdot[(\nabla - iM\mathbf{A})\psi] -(U/2)\,|\,\psi\,|^4 - (Mk/2)(\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{A}) \tag{1}$$

in which $\psi(\mathbf{x}, t)$ is the condensate field, and $A(\mathbf{x}, t)$ is a gauge field which is interpreted (up to a sign) as the depletion velocity. See also Chela-Flores (1977), Hu (1977), and Chela-Flores et al. (to be published). We shall call this last paper I.

The present note does not concern low-temperature physics, but rather the relevance of $\mathscr{L}^{(k)}_{(U)}$ to the foundations of field theory.

(i) Let us assume $k \neq 0$. We have shown in I the (1) implies via Noether's theorem the existence of a conserved current that is such that, when combined with Euler–Lagrange equations (ELE), it results that all solutions with nonstatic density $|\,\psi\,|^2$ are forbidden. This is true even if $U = 0$. Here a discontinuity in the Lagrangian parameters is exhibited because $\mathscr{L}^{(0)}_{(0)}$ is the Schrödinger Lagrangian, which certainly allows for nonstationary solutions.

(ii) Let us again assume $k \neq 0$. We observe that (1) is a singular Lagrangian, i.e., the canonical coordinates and momenta are restricted by constraints. Therefore Dirac mechanics applies (Dirac, 1950, 1964, 1966; Bergmann and Goldberg, 1955; Kundt, 1966; Marx, 1972; Sudarshan and Mukunda, 1974). We proved in I that there are no first-class constraints. But the canonical allowed variables (called observables by Marx, 1972), which are the state vector variables, are those that have weakly zero Poisson brackets with all first-class constraints. [See the above literature and also Section II of Kálnay and Tascón (1976).] Therefore, both fields $\psi(\mathbf{x}, t)$ and the gauge field $\mathbf{A}(\mathbf{x}, t)$ are (classical) state vector variables. This is in contrast with classical electrodynamics, where gauge transformations do not change the state.

(iii) Result (ii) is also valid for $\mathscr{L}^{(k)}_{(0)}$. However, this last Lagrangian is almost equal to that of Schrödinger field interacting with a nonquantized electromagnetic field [the case for which Dirac mechanics was applied by Marx (1972)]. Only the scalar potential and the electric field are missing. Let us assume that we add those missing terms to the Lagrangian, but each multiplied by a parameter k'. Then for $k' = 1$ we have the Schrödinger–Maxwell case and first-class constraints exist (Marx, 1972). (They also exist for all other values $k' \neq 0$.) But for $k' = 0$, it is clear that result (ii) holds: There are no first-class constraints. This is another example of discontinuities in the mathematical and physical properties which may be provoked by parameters that look harmless.

(iv) Concerning the physical role (ii) of gauge transformations we would like to mention two other cases which (although in a different context) share with (ii) the idea that gauge transformations could change physics. One is the work by Bohm and Aharonov (1959) concerning the significance of electromagnetic potentials in the quantum theory. The other example (Kálnay and Ruggeri, 1973) is that gauge transformations of finite mechanical systems (addition of a total time derivative to the Lagrangian) may change the physics if the Lagrangian is *singular*.

(v) Consider $k \neq 0$, U arbitrary and a solution of the ELE such that $\psi \not\equiv 0$, $A \equiv 0$. (The existence of such solutions is proved exhibiting examples.) One would guess that this solution could always have a corresponding solution of the ELE of the system whose Lagrangian density is

$$\mathscr{L}' \equiv \mathscr{L}^{(k)}_{(U)} \Big|_{\mathbf{A}\equiv 0} \tag{2a}$$

i.e.,

$$\mathscr{L}' = -(i/2)(\psi \partial_t \psi^* - \psi^* \partial_t \psi) - (1/2M)(\nabla\psi)^* \cdot (\nabla\psi) - (U/2) |\psi|^4 \tag{2b}$$

such that both solutions coincide. However, in the first case one obtains the ELE by variation of ψ, ψ^* and $\mathbf{A}$. The variation of $\mathbf{A}$ leads to

$$k\nabla \times (\nabla \times \mathbf{A}) + \mathbf{A} |\psi|^2 - (1/2Mi)(\psi^* \nabla\psi - \psi \nabla\psi^*) \approx 0 \tag{3}$$

In the second case we only subject to variation ψ and ψ^*. Therefore the ELE (3) is missing. However, going back to the first case, the solution $\psi \not\equiv 0$, $\mathbf{A} \equiv 0$

inserted in equation (3) reads

$$\psi^*\nabla\psi - \psi\nabla\psi^* \approx 0 \tag{4}$$

Equation (4) is a restriction among the ψ, ψ^* fields only, restriction which exists wherever the Lagrangian $\mathscr{L}_{(U)}^{(k)}$ has a solution with $\mathbf{A} = 0$. But this restriction is missing for the Lagrangian $\mathscr{L}'$ in spite of equation (2a) as if the system (1) would have "memory" of the gauge field even when it is zero!

References

Bergmann, P. G., and Goldberg, I. (1955). *Physical Review*, **98**, 531.

Bohm, D., and Aharonov, Y. (1959). *Physical Review*, **115**, 485.

Chela-Flores, J. (1975). *Journal of Low Temperature Physics*, **21**, 307.

Chela-Flores, J. (1977). *Journal of Low Temperature Physics*, **28**, 213.

Chela-Flores, J., Kálnay, A. J., Rodríguez-Gómez, J., Rodríguez-Nunez, J., and Tascón, R. (to be published) "Gauge Fields, Dirac Mechanics and Low Temperature Physics" and "Gauge Fields and the Hydrodynamics of Superfluid Helium Four". Also, preprints IVIC, Apdo. 1827, Caracas 101, to appear.

Dirac, P. A. M. (1950). *Canadian Journal of Mathematics*, **2**, 129.

Dirac, P. A. M. (1964). *Lectures on Quantum Mechanics*, Belfer Graduate School of Sciences Monograph Series No. 2, Yeshiva University, New York.

Dirac, P. A. M. (1966). *Lectures on Quantum Field Theory*, Belfer Graduate School of Sciences Monograph Series No. 3, Yeshiva University, New York.

Hu, B. (1977), "Relativistic Superconductivity and Superfluidity: A Unified View." Preprint No. A256.0177, Ecole Polytechnique, 91128 Palaiseau, France.

Kálnay, A. J., and Ruggeri, G. J. (1973). *International Journal of Theoretical Physics*, **8**, 189.

Kálnay, A. J., and Tascón, R. (1976). "On Lagrange, Hamilton-Dirac and Nambu Mechanics" (to be published). Also preprint IVIC, Apdo. 1827, Caracas 101, Venezuela.

Kundt, W. (1966). "Canonical Quantization of Gauge Invariant Field Theories," pp. 108–168 in Hoehler, G., ed., *Springer Tracts in Modern Physics*, Vol. 40. Springer-Verlag, Berlin.

Marx, E. (1972). *International Journal of Theoretical Physics*, **6**, 307.

Sudarshan, E. C. G., and Mukunda, N. (1974). *Classical Dynamics: A Modern Perspective.* John Wiley & Sons, New York.